תרגיל 5 יסודות תורת הפונקציות המרוכבות

(1)
$$\underline{\int_0^{2\pi}\frac{dt}{1-2a\cos t+a^2}} = \int_0^{2\pi}\frac{dt}{1+a^2-a(e^{it}+e^{-it})}\cdot\frac{ie^{it}}{ie^{it}} =$$

$$= -i\int_0^{2\pi}\frac{ie^{it}\,dt}{(1+a^2)e^{it}-a(e^{2it}+1)} = \left\{\begin{matrix}\gamma:[0,2\pi]\to\mathbb{C}\\ \gamma(t)=e^{it}\end{matrix}\right\} =$$

$$= -i\int_\gamma\frac{dz}{(1+a^2)z-a(z^2+1)} = i\int_\gamma\frac{dz}{az^2-(1+a^2)z+a} =$$

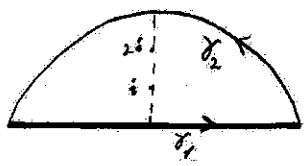
$$= i\int_\gamma\frac{dz}{(az-1)(z-a)} = i\int_\gamma\frac{1}{1-a^2}\left(\frac{a}{az-1}-\frac{1}{z-a}\right)dz =$$

$$= \frac{i}{1-a^2}\left(\int_\gamma\frac{dz}{z-\frac{1}{a}}-\int_\gamma\frac{dz}{z-a}\right) = \frac{i\cdot 2\pi i}{1-a^2}\cdot\left(n(\gamma,\tfrac{1}{a})-n(\gamma,a)\right) =$$

[נתון $|a|<1 \Rightarrow n(\gamma,a)=1\,,\ n(\gamma,\frac{1}{a})=0\ \ (|\frac{1}{a}|=\frac{1}{|a|}>1)$]

$$= -\frac{2\pi}{1-a^2}(0-1) = \underline{\frac{2\pi}{1-a^2}}$$

(2)



$$\gamma_1(t)=t\,,\quad \gamma_2(t)=Re^{it}$$

$$\underline{\int_\gamma\frac{dz}{(z^2+1)(z^2+4)}} = \frac{1}{3}\int_\gamma\left(\frac{1}{z^2+1}-\frac{1}{z^2+4}\right)dz = \frac{1}{3}\frac{1}{2i}\int_\gamma\left(\frac{1}{z-i}-\frac{1}{z+i}\right)dz -$$

$$-\frac{1}{3}\cdot\frac{1}{4i}\int_\gamma\left(\frac{1}{z-2i}-\frac{1}{z+2i}\right)dz = -\frac{i}{6}\cdot 2\pi i\,(n(\gamma,i)-n(\gamma,-i)) +$$

$$+\frac{i}{12}\cdot 2\pi i\,(n(\gamma,2i)-n(\gamma,-2i)) = \frac{\pi}{3}(1-0)-\frac{\pi}{6}(1-0) = \underline{\frac{\pi}{6}}$$

$$\left|\int_{\gamma_2}\frac{dz}{(z^2+1)(z^2+4)}\right| = \left|\int_{\gamma_2}\frac{dz}{z^4+5z^2+4}\right| = \left|\int_0^\pi\frac{iRe^{it}dt}{R^4e^{4it}+5R^2e^{2it}+4}\right| \le$$

$$\le \int_0^\pi\frac{R\,dt}{|R^4e^{4it}+5R^2e^{2it}+4|} \overset{(*)}{\le} \int_0^\pi\frac{R\,dt}{R^4-5R^2-4} = \underline{\frac{\pi R}{R^4-5R^2-4}\xrightarrow[R\to\infty]{}0}$$

$$(*):\ |R^4e^{4it}+5R^2e^{2it}+4| \ge |R^4e^{4it}|-|5R^2e^{2it}|-4 =$$

$$= R^4-5R^2-4$$ לפי אי-שוויון המשולש.

$$\underline{\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)(x^2+4)}} = \int_{\gamma_1}\frac{dz}{(z^2+1)(z^2+4)} = \int_\gamma f\,dz-\int_{\gamma_2}f\,dz = \frac{\pi}{6}-0 = \underline{\frac{\pi}{6}}$$

(3)
$$\int_\gamma\frac{e^{iz}}{z^2+a^2}dz = \frac{1}{2ia}\left(\int_\gamma\frac{e^{iz}}{z-ia}dz-\int_\gamma\frac{e^{iz}}{z+ia}dz\right)$$

γ מסילה סגורה ולא מקיפה את $-ia$ לכן $\frac{e^{iz}}{z+ia}$

אנליטית במרחב המכיל את γ ולכן $\int_\gamma\frac{e^{iz}}{z+ia}dz=0$

לכן $\underline{\int_\gamma\frac{e^{iz}}{z^2+a^2}dz = \frac{1}{2ia}\int_\gamma\frac{e^{iz}}{z-ia}dz}$,

$$e^{iz}=1+\sum_{n=0}^{\infty}\frac{(iz)^n}{n!} \Rightarrow e^{i(z-ia)}=1+\sum_{n=0}^{\infty}\frac{i^n}{n!}(z-ia)^n$$

לכן $\frac{e^{i(z-ia)}-1}{z-ia}=\frac{\sum_{n=1}^{\infty}\frac{i^n}{n!}(z-ia)^n}{z-ia}=\sum_{n=1}^{\infty}\frac{i^n}{n!}(z-ia)^{n-1}$ אנליטית,

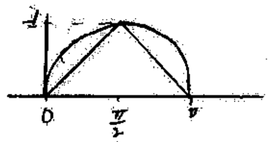
כלומר $\frac{e^{iz+a}-1}{z-ia}$ אנליטית ולכן $\int_\gamma \frac{e^{iz+a}-1}{z-ia}\,dz=0$

$$\int_\gamma \frac{e^{iz}}{z^2+a^2}\,dz=\frac{1}{2ia}\int_\gamma \frac{e^{iz}}{z-ia}\,dz=\frac{e^{-a}}{2ia}\left(\overbrace{\int_\gamma \frac{e^{iz+a}-1}{z-ia}\,dz}^{0}+\int_\gamma \frac{dz}{z-ia}\right)=$$

$$=\frac{e^{-a}}{2ia}\int_\gamma \frac{dz}{z-ia}=\frac{e^{-a}}{2ia}\cdot 2\pi i\cdot n(\gamma,ia)=\frac{\pi}{ae^a}$$

$$\left|\int_{\gamma_2}\frac{e^{iz}}{z^2+a^2}\,dz\right|=\left|\int_0^\pi \frac{e^{iRe^{it}}}{R^2e^{2it}+a^2}\cdot iRe^{it}\,dt\right|\le\int_0^\pi \frac{R\cdot e^{-R\sin t}}{|R^2e^{2it}+a^2|}\,dt\le$$

$$\le\int_0^\pi \frac{Re^{-R\sin t}}{R^2-a^2}\,dt \;\le$$



לכל $0\le x\le\frac{\pi}{2}$, $\sin x\ge\frac{2}{\pi}x$ ולכל $\frac{\pi}{2}<x\le\pi$, $\sin x\ge 2-\frac{2}{\pi}x$ לכן:

$$\le\frac{R}{R^2-a^2}\left(\int_0^{\frac{\pi}{2}}e^{-\frac{2R}{\pi}t}\,dt+\int_{\frac{\pi}{2}}^{\pi}e^{-R(2-\frac{2}{\pi}t)}\,dt\right)=$$

$$=\frac{R}{R^2-a^2}\left[-\frac{\pi}{2R}(e^{-R}-1)+e^{-2R}\cdot\frac{\pi}{2R}(e^{2R}-e^{R})\right]=$$

$$=\frac{\pi}{R^2-a^2}\cdot\frac{1}{2}\left(-e^{-R}+1+1-e^{-R}\right)=\frac{\pi(1-\overbrace{e^{-R}}^{\to 0})}{R^2-a^2}\xrightarrow[R\to\infty]{}0$$

$$\int_{-\infty}^{\infty}\frac{\cos x}{x^2+a^2}\,dx=\int_{\gamma_1}\frac{\cos z}{z^2+a^2}\,dz=Re\int_{\gamma_1}\frac{e^{iz}}{z^2+a^2}\,dz=$$

$$=Re\left(\int_\gamma-\int_{\gamma_2}\right)=Re\left(\frac{\pi}{ae^a}-0\right)=\frac{\pi}{ae^a}$$

$$\int_{-\infty}^{\infty}\frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx=\frac{1}{a^2-b^2}\int_{-\infty}^{\infty}\left(\frac{1}{x^2+b^2}-\frac{1}{x^2+a^2}\right)\cos x\,dx \quad (4)$$

$$\underset{(3)}{\overset{\text{לפי}}{=}}\;\frac{1}{a^2-b^2}\cdot\left(\frac{\pi}{be^b}-\frac{\pi}{ae^a}\right)$$